KdV-hierarchy and soliton solutions

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Abstract

Our aim is to give a compact and consistent overview of the steps leading to the discovery of the first infinite dimensional completely integrable Hamiltonian system. We construct the Lax pair for the Korteweg-de Vries equation and obtain the KdV-hierarchy using two different approaches. The Inverse Scattering Transform as a method for solving the KdV-equation is introduced, and N-soliton solutions for this equation are discussed qualitatively. By constructing an infinite series of first integrals, the Hamiltonian structure of the KdV-hierarchy is revealed.

1 Introduction

In 1844, the Scottish naval engineer John Scott Russell firstly described solitary waves [1], i.e waves which travel long distances without changing their shape. After becoming aware of these waves during experiments on ship construction, he started to investigate this phenomenon in water tanks. An appropriate theoretical explanation was found no less than fifty years later by Korteweg and de Vries [2], who formulated their famous equation[1]

$$u_t = 6uu_x - u_{xxx}. \tag{1}$$

There was no noteworthy research in this area for a long time, until Fermi, Pasta and Ulam firstly did numerical simulations on vibrating strings containing nonlinear terms in 1954. Surprisingly, they found a quasi-periodic behaviour in many cases instead of the chaotic one they actually expected [3]. When Kruskal and Zabusky found links between these results and the KdV-solitons in new numerical simulations eleven years later [4], the interest in equation (1) was suddenly revived.

We review some of the pioneering discoveries triggered by this observation. The discovery of a whole hierarchy of nonlinear PDEs called the KdV-hierarchy, the Inverse Scattering Transform (IST) as a method for solving equation (1) and the solution for N-solitons in closed form are among these. The Hamiltonian interpretation of the KdV-hierarchy is discussed. Ultimately, we sketch the proof that this is indeed a completly integrable system which became one of the pillars on the way to the modern theory of classical integrable systems.

[1]There are many different conventions about signs and numerical factors for this equation, resulting from scaling. In the following, we will persistently use the version stated above.

2 Steps towards the KdV hierarchy

Two remarkable discoveries. When a team around Gardner [5] searched for a solution to eq. (1) in 1967, they found the following interesting approach inspired by the Cole-Hopf-transformation for Burger's equation: By solving the eigenvalue equation for the Schrödinger operator[2]

$$\Big(\partial^2 - u(x,t)\Big)\Psi(x,t) = -\lambda(t)\Psi(x,t)$$

for u and substituting this in eq. (1), one obtains

$$\lambda_t \Psi^2 + \partial\Big(\Psi Q_x - \Psi_x Q\Big) = 0, \tag{2}$$

where $Q \equiv \Psi_t + \Psi_{xxx} - 3(u-\lambda)\Psi_x$. Since $u(x,t)$ depends on time, Ψ and λ are parametrically time-dependent, too. Assuming that $\Psi \to 0$ for $|x| \to \infty$, integration over $(-\infty, +\infty)$ yields

$$\lambda_t \int_{-\infty}^{+\infty} \Psi^2 dx = 0 \quad \Rightarrow \quad \lambda = \text{const.}$$

Consequently, the spectrum of the Schrödinger operator is at least partially preserved in time if u evolutes according to eq. (1). This discovery led to the invention of the Inverse Scattering Transform which will be considered in Section 3.

One year later, Lax [7] formulated his famous equation describing isospectral evolution, which is derived as follows: Consider a PDE which is nonlinear in $\frac{\partial}{\partial x} \equiv \partial$, but first order in $\frac{\partial}{\partial t}$ of the form

$$u_t = K[u]. \tag{3}$$

If we can associate to each u a selfadjoint and therefore diagonalizable operator L_u such that $L(t)$ remains unitarily equivalent,

$$\frac{\mathrm{d}}{\mathrm{dt}}(U(t)^{-1}\, L(t)\, U(t)) = -U^{-1}\dot{U}U^{-1}LU + U^{-1}\dot{L}U + U^{-1}L\dot{U} = 0,$$

then *the eigenvalues of* L *are constants of motion.* Since the one-parameter group $U(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{dt}}U(t) = M\,U(t)$ with M being antisymmetric, one gets

$$\frac{\mathrm{d}}{\mathrm{dt}}L \equiv \dot{L} = [M, L]. \tag{4}$$

This is the Lax equation which has to be equivalent to the original equation (3). The advantage of this method is that the *whole spectrum* of L is preserved!

Lax pair of the KdV equation and generalizations. Clearly, we are interested in the Lax pair of eq. (1). The partly isospectral behaviour of the Schrödinger operator strongly suggests to set $L = \partial^2 - u(x,t)$. Since then $\dot{L}$ acts like $-u_t$, we are left with finding a suitable, antisymmetric operator M which satisfies

$$[\partial^2 - u\,,\, M] = 6uu_x - u_{xxx}$$

in order to recast eq. (1) from eq. (4). The simplest candidate for M is the differential operator ∂ itself, which is antisymmetric with respect to the L^2-inner product, $\partial^* = -\partial$. This yields a chiral wave equation

$$[\partial^2 - u, \partial] = u_x \quad \Rightarrow \quad u_x = u_t.$$

[2]Since the symbol D is used for the Hirota-D-operator in the theory of integrable systems, we denote the differential operator with respect to x by ∂ in the following, using the same notation as [6].

This is not the equation we desire, however, this is in fact the zeroth order of the KdV-hierarchy. As we will come back to this equation later on, we set $M_0 = -4\partial$ and $u_x = -\frac{1}{4}K_0[u]$. Trying a general third-order differential operator $\partial^3 + b(x)\partial + \partial b(x)$ instead, we get

$$[\partial^2 - u\,,\, \partial^3 + b\partial + \partial b] = u_{xxx} + b_{xxx} + 2bu_x + (4b_x + 3u_x)(\partial^2 + \partial).$$

As the KdV-equation is of order zero in ∂, the factor in front of the rightmost bracket has to vanish; thus we set $b = -\frac{3}{4}u$ which yields

$$[\partial^2 - u\,,\, \partial^3 - \frac{3}{4}(u\partial + \partial u)] = \frac{1}{4}u_{xxx} - \frac{6}{4}uu_x = -\frac{1}{4}K_1[u].$$

We found the right result up to a factor and therefore, the Lax pair for the KdV equation is given by

$$L = \partial^2 - u \qquad \text{and} \qquad M_1 = -4\partial^3 + 3(u\partial + \partial u). \tag{5}$$

This principle can clearly be generalized and results in an infinite series of nonlinear PDEs, the KdV-hierarchy. Using antisymmetric operators

$$M_n = -4\Big(\partial + \sum_{i=1}^{n}(b_i\partial^{2i-1} + \partial^{2i-1}b_i)\Big)$$

of arbitrary order n and requiring that $[\partial^2 - u\,,\, M_n]$ is of order zero in ∂ in order to fix the b_i, we can write the KdV$_n$ equation

$$[\partial^2 - u\,,\, M_n] = K_n[u] = u_t. \tag{6}$$

As an example, we provide the Lax matrix M_2 and KdV$_2$:

$$M_2 = -4\partial^5 + 5(u\partial^3 + \partial^3 u) - \frac{5}{4}(u_{xx}\partial + \partial u_{xx}) - \frac{15}{4}(u^2\partial + \partial u^2)$$

$$4u_t = -30u_x u^2 + 20u_{xx}u_x + 10u_{xxx}u - u_{xxxxx}$$

In this case, however, two unknown functions b_1 and b_2 have to meet four equations already. Like in the case when $n = 1$, there are two linearly dependent equations. But the third equation turns out to be the derivative of the fourth. Consequently, it is far from obvious whether the number of unknown functions b_i matches with the number of constraints in general.

Representation by pseudo-differential operators. A way to describe the KdV-hierarchy using fractional powers of the Schrödinger operator was found in 1976 by Gelfand and Dikii [8]. A more recent description of this method is given in [6]. First, extend the usual differential operator ∂ obtaining the Leibnitz rule $[\partial, f] = (\partial f) \equiv f_x$ by defining an "integration" symbol ∂^{-1} such that

$$\partial^{-1}\partial = \partial\partial^{-1} = 1, \qquad \text{where} \qquad \partial^{-1}f = \sum_{i=0}^{\infty}(-1)^i(\partial^i f)\,\partial^{-i-1}.$$

The relation on the right-hand side is motivated by integration by parts and can be generalized for ∂^{-k}. This allows us to shuffle all the ∂^{-k} to the right in concatenations. A general pseudo-differential operator $A = \sum_{i=-\infty}^{N} a_i(x)\partial^i$ can be split into two parts, $A = (A)_+ + (A)_-$, where

$$(A)_+ = \sum_{i=0}^{N} a_i(x)\partial^i \quad \text{and} \quad (A)_- = \sum_{i=-\infty}^{-1} a_i(x)\partial^i.$$

With these tools, we can compute $L^{\frac{1}{2}}$ by formally writing $Q^2 = L \equiv \partial^2 - u$ and making a general ansatz $Q = \partial + \sum_{i=0}^{\infty} q_{-i}\partial^{-i}$. Next, Q^2 is calculated explicitely by multiplying the sum by itself and then pushing all (pseudo-)differentials to the right. Since $(L)_- = 0$, all coefficents of the negative powers of ∂ have to vanish in this expression, and one can determine the q_{-i} by recursion:

$$Q = \partial - \frac{1}{2}u\partial^{-1} + \frac{1}{4}u_x\partial^{-2} - \frac{1}{8}(u_{xx} + u^2)\partial^{-3} + \frac{1}{16}[u_{xx} + 3u^2]_x\partial^{-4} + \cdots$$

This enables us now to calculate $L^{\frac{n}{2}}$. Surprisingly, it turns out that

$$\begin{aligned}
(L^{\frac{1}{2}})_+ &= \quad (Q)_+ = \partial & &= -\frac{1}{4}M_0 \\
(L^{\frac{3}{2}})_+ &= \ (QL)_+ = \partial^3 - \frac{3}{4}(u\partial + \partial u) & &= -\frac{1}{4}M_1 \\
(L^{\frac{5}{2}})_+ &= (QL^2)_+ = \partial^5 - \frac{5}{4}(u\partial^3 + \partial^3 u) + \frac{5}{16}(u_{xx}\partial + \partial u_{xx}) + \frac{15}{16}(u^2\partial + \partial u^2) & &= -\frac{1}{4}M_2
\end{aligned}$$

hold. More generally, it can be shown that the KdV_n equation for $n \in \mathbb{N}_0$ is given by

$$\dot{L} = [\, L \,,\, 4(L^{\frac{2n+1}{2}})_+ \,] \equiv -K_n[u]. \tag{7}$$

A proof can be found in [6]. Note that for integer powers of L, $L^n = (L^n)_+$ holds, and therefore the equations become trivial.

3 Inverse scattering method and soliton solutions

Time-independent scattering. In 1968, Gardner, Greene, Kruskal and Miura [5] developed IST as a revolutionary method to solve eq. (1). In order to get an insight to this method, a static scattering problem with u being the potential is considered firstly, where the eigenvalue equation for the Schrödinger operator reads

$$L\Psi(x) \equiv \big(\partial^2 - u(x)\big)\Psi(x) = -\lambda\Psi(x).$$

The spectrum of L is twofold, with a continuous part for $\lambda > 0$ and a finite number of discrete negative eigenvalues λ_n corresponding to bound states. Assuming that u vanishes for large values of $|x|$, or, more formal, $\int_{-\infty}^{\infty}(1 + |x|)|u(x)|dx < \infty$, this reduces to

$$\partial^2\Psi(x) \approx -\lambda$$

for large $|x|$. This makes it possible to easily determine the asymptotic behaviour of the eigenfunctions $\Psi(x, k)$ and $\Psi_n(x)$ as illustrated in Figure 1. The scattering data s characterizes the asymptotic behavior of the eigenfunctions and consists of the transmission coefficent $a(k)$, the reflection coefficient $b(k)$, the discrete eigenvalues κ_n and the normalization coefficients c_n for the Ψ_n. We choose the normalization such that $\int \Psi_n^2\, dx = 1$ and $|a|^2 + |b|^2 = 1$.

It is more than a notable fact that there is a bijective mapping $u(x) \to s$ of potentials u into the scattering data $s = (b(k), \kappa_n, c_n)$. In particular, one can compute u from s by solving for $\mathcal{K}(x, y)$ in the Gelfand-Levitan integral equation (see for example [9])

$$\mathcal{K}(x, y) + B(x + y) + \int_x^{+\infty} \mathcal{K}(x, s)B(y + s)\, ds = 0, \tag{8}$$

$$\text{where} \qquad B(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k)e^{ik\xi}\, dk + \sum_{n=0}^{\infty} c_n^2 e^{-\kappa_n\xi}.$$

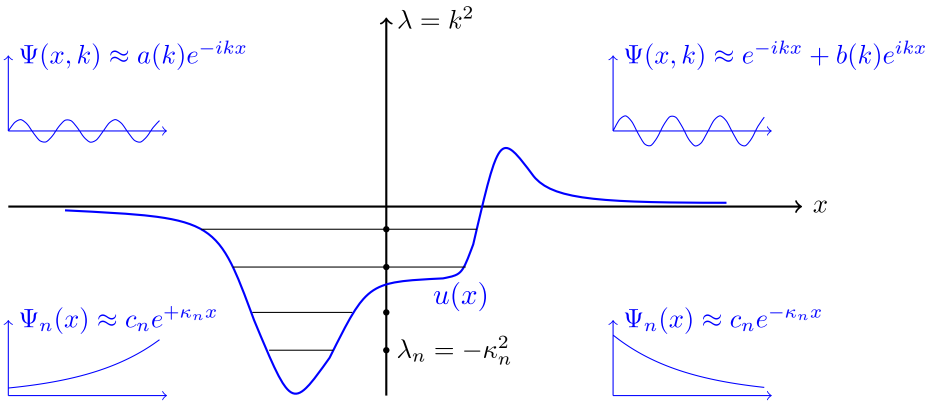


Figure 1: The time-independent scattering problem. The small graphs show the asymptotic behaviour of the eigenfunctions $\Psi_n(x)$ and $\Psi(x,k)$ in the corresponding regions. For $\lambda < 0$, only exponentially decaying solutions are feasible in order to be normalizable. For $\lambda > 0$, we consider an incoming wave from the right whose amplitude is set to 1.

The kernel B contains the scattering data s. In the derivation of this equation, the kernel $\mathcal{K}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_k(x)e^{-iky}\,dk - \delta(x-y)$ for $y \geq x$ originates from Fourier transforming Ψ and therefore obeys the wave equation $K_{xx}(x,y) = K_{yy}(x,y)$. Then, the potential is given by

$$u(x) = -2\frac{d}{dx}\mathcal{K}(x,x).$$

Inverse scattering transform. However, our aim is to solve eq. (1) and in this case, the potential $u(x,t)$ is time-dependent, and so is the scattering data except for $\kappa_n = \sqrt{-\lambda_n}$

$$s = (b(k,t), \kappa_n, c_n(t)).$$

As discussed more in detail in [5], it turns out that the time evolution of s can be obtained from eq. (2). Using the fact that $\lambda_t = 0$, this equation can be integrated yielding

$$\Psi_t + \Psi_{xxx} - 3(u-\lambda)\Psi_x = C(t)\Psi + D(t)\Psi\int_{-\infty}^{x}\frac{dx}{\Psi^2}.$$

Here, $C(t)$ and $D(t)$ are integration constants, and the rightmost term represents the second, linearly independent solution to eq. (2) caused by the twofold spectrum. Inserting the expressions for $\Psi_n(x)$ and $\Psi(x,k)$ into this equation, where both cases have to be treated separately, and using the normalization conditions, one finally obtains:

$$c_n(t) = c_n(0)e^{4\kappa_n^3 t} \qquad b(k,t) = b(k,0)e^{8ik^3 t} \qquad a(k,t) = a(k,0)$$

Note in particular that a and $|b|$ do not evolute in time.

Now the different parts introduced in this section can be put together, enabling us to solve the initial-value problem $u_t = 6uu_x - u_{xxx}$ with $u(x,0) = u_0(x)$ for $t > 0$ by the Inverse Scattering Transform: First, the scattering data s is obtained from the initial data $u_0(x)$ by solving the direct scattering problem. By analyzing how s evolutes in time, we retrieve time-dependent scattering data. Finally, $u(x,t)$ can be computed from the solution of eq. (8), where both the Gelfand-Levitan equation and its solution are now parametrically time-dependent. The procedure is illustrated in Figure 2. In particular, there exists a unique solution for every $u_0(x)$ vanishing sufficiently fast for $|x| \to \infty$. Nevertheless, it is noted that one can easily guess a solution of eq. (1) which does not meet the boundary conditions, for instance $u(x,t) = -x/6t$.

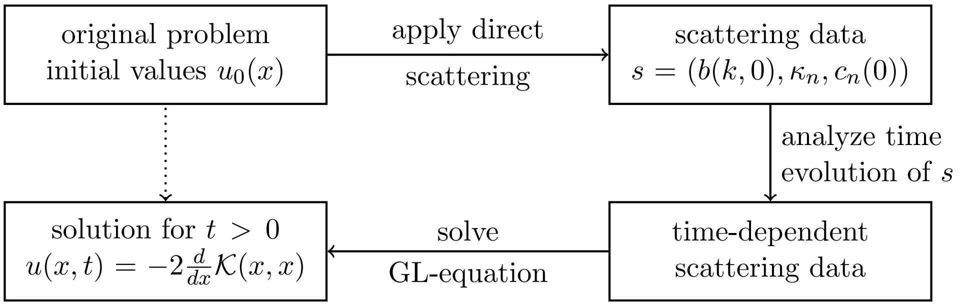


Figure 2: Schematic illustration of the inverse scattering transform

Soliton solutions. An explicit solution describing the interactions of N solitons was found independently by Hirota in 1971 [10]. However, the same solution can of course also be derived through the IST, see for example [9]. This solution corresponds to a reflectionless potential in the Schrödinger equation, i.e., $b(k,t) \equiv 0$ and is given by

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \ln \det |M(x,t)|,$$

$$\text{where} \quad M_{ij}(x,t) = \delta_{ij} + \frac{2\sqrt{\kappa_i \kappa_j}}{\kappa_i + \kappa_j} e^{\xi_i + \xi_j} \quad \text{and} \quad \xi_i = \kappa_i x - 4\kappa_i^3 t - \ln c_i(0).$$

The dimension of the quadratic matrix M corresponds to the number N of solitons involved. The κ_n determine speed and amplitude, the $c_n(0)$ the initial position and they have to be mutually different. For $N = 1$, we find the well-known solution [11]

$$u_1(x,t) = -2\kappa^2 \operatorname{sech}^2(\kappa x - 4\kappa^3 t - \ln c(0)).$$

Examples for two and three solitons are depicted in Figure 3, illustrating some basic properties. The speed of a soliton is higher, the larger its amplitude. The solitons do not simply pass through each other, but show nonlinear interaction processes. This interaction corresponds to a phase shift for $t \to \pm\infty$.

4 Conserved quantities and Hamiltonian structures

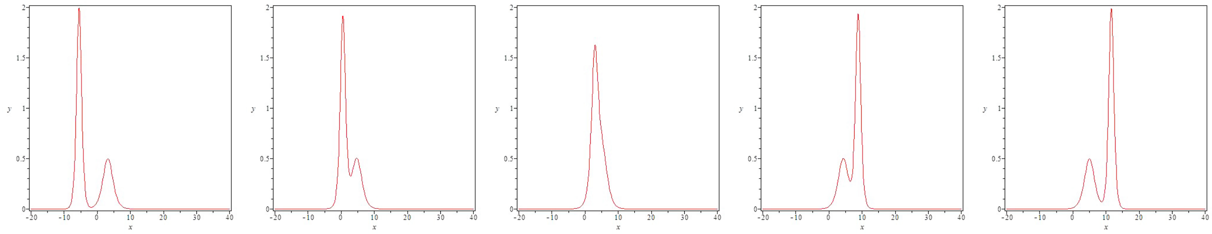
First Integrals. A simple way to make the Hamiltonian structure of the KdV-equation perceivable is to construct an infinite series of so-called first integrals, following [12]. A one-dimensional local conservation law between a density T and a flux X takes the form

$$T_t[u] - X_x[u] = 0. \tag{9}$$

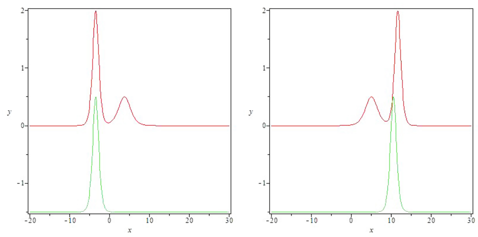
This conservation law is called local, since T and X are *not*[3] explicitly dependent on x and t. In our case, they will turn out to be polynomials in u and its derivatives. Then one can define the time-independent first integrals $I^{(n)}$ arising from these laws as

$$I^{(n)} = \int_{-\infty}^{+\infty} T^{(n)}[u]\, dx \qquad \text{where} \qquad \frac{d}{dt} I^{(n)} = X^{(n)}\Big|_{-\infty}^{+\infty} = 0,$$

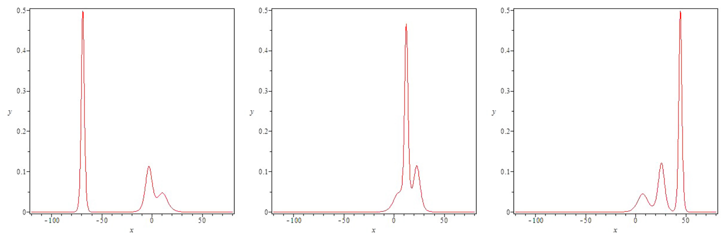
[3] In physics, the term "local" is frequently used in the opposite sense. However, here it is used as defined by Gardner in [12].



(a) Interaction of two solitons



(b) Phase shift



(c) Interaction of three solitons

Figure 3: N soliton solutions according to the Hirota solution, visualized using Maple© 14. Note that $-u(x,t)$ is plotted here for selected values of t. Part (a) shows nonlinear interaction between two solitons. Part (b) depicts the same two solitons, and additionally a single soliton (in green) with the same initial data as one of them. Clearly, the interaction results in a phase shift. Part (c) illustrates a three soliton solution.

using eq. (9) and still assuming that $u \to 0$ for $|x| \to \infty$. For example, the KdV_1-equation itself can be written as a conservation law, enabling us to read off the associated density and first integral

$$u_t - (3u^2 - u_{xx})_x = 0 \qquad \Rightarrow \qquad I^{(1)} = \int u\,dx.$$

It should be noted that not every constant of motion has an associated conservation law. As a next step, we apply the following useful transformation found by Miura [13]

$$u = w + \varepsilon w_x + \varepsilon^2 w^2 \tag{10}$$

on the KdV equation which yields

$$(1 + \varepsilon\partial + 2\epsilon^2 w)\Big(w_t - \big[3w^2 + 2\varepsilon^2 w^3 - w_{xx}\big]_x\Big) = 0.$$

The expression on the left-hand side is an operator and therefore cannot vanish, but the expression on the right-hand side is clearly a conservation law. This tells us that $\int w\,dx$ is a first integral! Eq. (10) is an equation of the Ricatti type which can be solved by plugging in a formal power series ansatz $w = \sum_{n=0}^{\infty} w_n \varepsilon^n$ and equating powers of ε. We get a recursion relation

$$-w_n = w'_{n-1} + \sum_{p=0}^{n-2} w_{n-p} w_p, \qquad w_0 = u, \qquad w_1 = -u',$$

which enables us to express w by u and its derivatives. On integration, all perfect derivatives vanish, and one finally obtains

$$\int w\,dx = \underbrace{\int u\,dx}_{I^{(1)}} - \varepsilon^2 \underbrace{\int u^2\,dx}_{I^{(2)}} + \varepsilon^4 \underbrace{\int 2u^3 + u_x^2\,dx}_{I^{(3)}} + \cdots \tag{11}$$

Since ε can be chosen arbitrarily, each term in eq. (11) constitutes a first integral.

KdV as a completely integrable system. Looking closer at $I^{(3)}$ in eq. (11), one sees that

$$\partial\Big(\frac{\delta I^{(3)}[u]}{\delta u}\Big) = 2(6uu_x - u_{xxx}) = 2K_1[u].$$

Similar relations exist between all $K[u]_n$ and the $I^{(n)}[u]$ which was realized by Gardner as acknowledged in a footnote by Lax [7]. Defining the skew symmetric operator $\mathcal{J} := \partial$ and setting $H = \frac{1}{2}I^{(3)}$, we retrieve

$$u_t = \mathcal{J}\,\frac{\delta H[u]}{\delta u}. \tag{12}$$

Thus, we can rewrite the KdV-equation as Hamiltonian equation of motion, where u takes over the role of the generalized coordinates labeled by x! An infinite-dimensional Poisson bracket can be obtained applying the chain rule to a functional $F[u]$ which is not explicitly dependent on t. Here, $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product with respect to x

$$\frac{dF}{dt} = \Big\langle \frac{\delta F}{\delta u}, \frac{du}{dt}\Big\rangle = \Big\langle \frac{\delta F}{\delta u}, \mathcal{J}\frac{\delta H}{\delta u}\Big\rangle =: \{F, H\}\,.$$

Finally, in 1971, Zakharov and Faddeev succeeded in showing that the IST is indeed a canonical transformation to action-angle variables [14]. As the proof is quite extensive, we only sketch the main steps here:

1. Establish a "symplectic form" ω in the generalized coordinates $u(x)$

$$\omega = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} \delta_1 u(x)\delta_2 u(y) - \delta_2 u(x)\delta_1 u(y)\,dy\,dx.$$

2. Express it in new coordinates which depend merely on the scattering data

$$\omega = \int_{-\infty}^{+\infty} \delta_1 P(k)\delta_2 Q(k) - \delta_2 P(k)\delta_1 Q(k)\,dk + \sum_{l=1}^{m}(\delta_1\tilde{P}_l\delta_2\tilde{Q}_l - \delta_1\tilde{Q}_l\delta_2\tilde{P}_l),$$

   where $P(k), \tilde{P}(\kappa_n)$ are constants of motion, i.e. functions of $|b(k)|$ and κ_n only.

3. Show that the Hamiltonian $H = I^{(3)}$ is a function of the impulses $P(k)$ only.

This can be considered as a generalization of Liouville integrability with functional analysis means and clearly shows that the KdV hierarchy is an infinite-dimensional completely integrable Hamiltonian system.

5 Outlook

Once the facts we presented were recognized, many other scientists became interested in this subject, and an overwhelming flood of publications started. Related series of nonlinear PDEs like the Generalized KdV-hierarchy and the Kadomtsev-Petviashvili hierarchy were discovered. By exploring the algebraic and geometric structures behind them, powerful tools like the Adler trace for pseudo-differential operators were developed. The Inverse Scattering Transform was transferred to the quantized case, and more recently, a connection to conformal field theory was found.

We can conclude that the discovery of the KdV-hierarchy was a groundbreaking step towards the modern theory of classical integrable systems. Moreover, without any doubt, it also influenced many other comtemporary areas of research both in mathematics and theoretical physics.

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